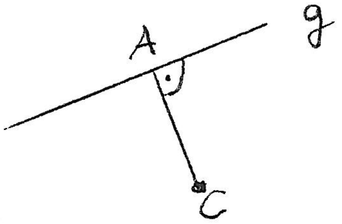
Lösungen

1.a)



$A(3+2t \mid 6+3t \mid -10-6t)$

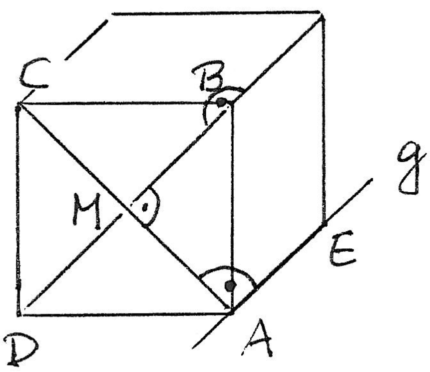
$\vec{CA} = \begin{pmatrix} 1+2t \\ -2+3t \\ -17-6t \end{pmatrix} \qquad \vec{CA} \circ \vec{g} = 0$

$\Rightarrow 2+4t-6+9t+102+36t = 0$

$\underline{\underline{A(-1 \mid 0 \mid 2)}} \qquad \Rightarrow 49t = -98 \qquad t = -2$

$|\vec{AC}| = \sqrt{9+64+25} = \sqrt{98} = \underline{\underline{7 \cdot \sqrt{2}}} \qquad \vec{AC} = \begin{pmatrix} 3 \\ 8 \\ 5 \end{pmatrix}$

b)



Kantenlänge $\underline{\underline{K = 7}}$

$M(0.5 \mid 4 \mid 4.5)$

$\vec{BD} \perp \vec{g}$ und $\vec{BD} \perp \vec{AC}$

$\vec{g} \times \vec{AC} = \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} \times \begin{pmatrix} 3 \\ 8 \\ 5 \end{pmatrix} = \begin{pmatrix} 63 \\ -28 \\ 7 \end{pmatrix}$

$\vec{BD} = \begin{pmatrix} 9 \\ -4 \\ 1 \end{pmatrix}$ mit $|\vec{BD}| = \sqrt{98}$ $\qquad [C(2 \mid 8 \mid 7)]$

$\vec{D} = \vec{M} + \frac{1}{2} \cdot \vec{BD} = \begin{pmatrix} 0.5 \\ 4 \\ 4.5 \end{pmatrix} + \begin{pmatrix} 4.5 \\ -2 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$ $\qquad \underline{\underline{D(5 \mid 2 \mid 5)}}$

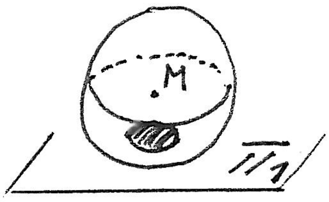
$\underline{\underline{B(-4 \mid 6 \mid 4)}}$

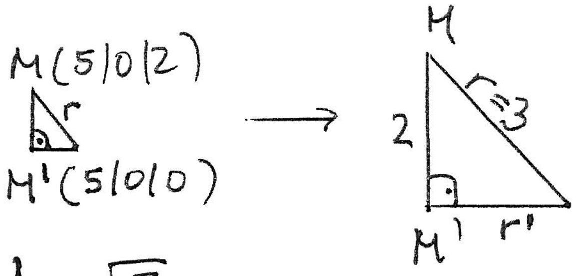
c) $\vec{AE} = \pm \vec{g} = \pm \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$ mit $|\vec{AE}| = 7$

Variante 1: $E(1 \mid 3 \mid -4)$, $F(-2 \mid 9 \mid -2)$
$H(7 \mid 5 \mid -1)$, $G(4 \mid 11 \mid 1)$

Variante 2: $E^*(-3 \mid -3 \mid 8)$, $F^*(-6 \mid 3 \mid 10)$
$H^*(3 \mid -1 \mid 11)$, $G^*(0 \mid 5 \mid 13)$

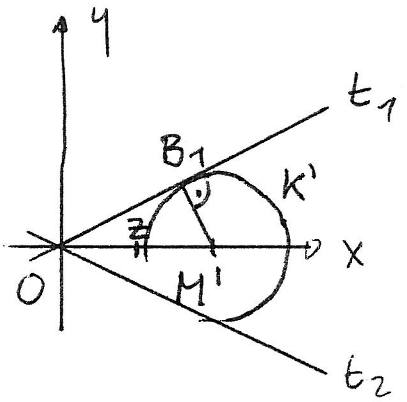
2 a) Kugel : $M(5|0|2)$, $r=3$





$\underline{\underline{M'(5|0|0)}}$ $\qquad$ $\underline{\underline{r' = \sqrt{5}}}$

b)



Kreis k' in $\mathbb{R}^2$: $(x-5)^2+y^2=5$

Thaleskreis über OM':

$Z(2.5|0)$, $r=2.5$

$$\begin{array}{r|rcl|l} - & (x-2.5)^2+y^2 & = & 6.25 & \text{Th.kr.} \\ + & (x-5)^2+y^2 & = & 5 & k' \end{array}$$

$\Rightarrow -10x+25+5x-6.25=-1.25$

$20 = 5x \quad \Rightarrow x=4$

Berührungspunkte

$B_1(4|2)$, $B_2(4|-2)$ bzw. in $\mathbb{R}^3$: $B_1(4|2|0)$

$B_2(4|-2|0)$

Kugeltangenten:

$\underline{\underline{t_1: \vec{x} = \begin{pmatrix}0\\0\\0\end{pmatrix} + t\begin{pmatrix}2\\1\\0\end{pmatrix}}}$ $\qquad$ $\underline{\underline{t_2: \vec{x} = \begin{pmatrix}0\\0\\0\end{pmatrix} + t\begin{pmatrix}2\\-1\\0\end{pmatrix}}}$

3) $w = f(z) = \frac{2i}{z}$

a) Fixpunkte: $f(z) = z \Rightarrow \frac{2i}{z} = z$

$z^2 = 2i = 2 \cdot \text{cis}\,(90°)$

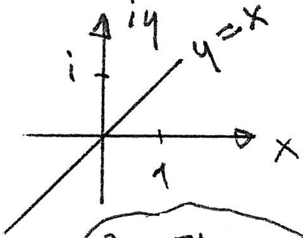
$z_1 = \sqrt{2} \cdot \text{cis}\,45° = 1+i$

$\underline{\underline{z_2}} = \sqrt{2} \cdot \text{cis}\,225° = \underline{\underline{-1-i}}$

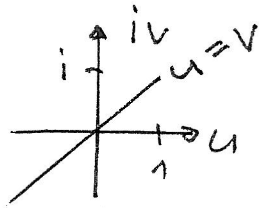
b) $w = u + iv = \frac{2i}{x+iy} \overset{V200}{=} \frac{2y}{x^2+y^2} + \frac{2x}{x^2+y^2} \cdot i$

Umkehrfunktion: $z = f^{-1}(w) = \frac{2i}{w}$

$z = x + iy = \frac{2i}{u+iv} = \frac{2v}{u^2+v^2} + i \cdot \frac{2u}{u^2+v^2}$



z-Ebene



w-Ebene

Gerade durch 0

$\Updownarrow$

Gerade durch 0

$y = x \quad \rightsquigarrow \quad \frac{2u}{u^2+v^2} = \frac{2v}{u^2+v^2} \Rightarrow \underline{\underline{u = v}}$

(Fixgerade)

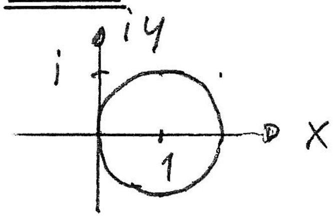
c) Gleichung der Gerade $v=1$ der w-Ebene

$v = 1 \quad \rightsquigarrow \quad \frac{2x}{x^2+y^2} = 1$ in der z-Ebene

$2x = x^2 + y^2$

$x^2 - 2x + 1 + y^2 = 1$

$\underline{\underline{\text{Kreis}}}$ $(x-1)^2 + y^2 = 1 \qquad M(1|0),\ r = 1$



z-Ebene

4. $\frac{y'}{y'+y} + e^x = 0 \Rightarrow y' + y' \cdot e^x + y \cdot e^x = 0$

$y'(1+e^x) + y \cdot e^x = 0 \quad [1+e^x > 0]$

$y' + \frac{e^x}{1+e^x} y = 0$

Trennung der Variablen

$\frac{dy}{dx} = -\frac{e^x}{1+e^x} y \Rightarrow \frac{1}{y} dy = -\frac{e^x}{1+e^x} dx$

Integration liefert (Substitution oder TR)

$\ln|y| = -\ln(1+e^x) + C \quad \Big| e^{\square}$
$[C \in \mathbb{R}]$

$\Rightarrow |y| = \frac{1}{1+e^x} \cdot e^C$

$\Rightarrow \underline{\underline{y = f(x) = \frac{K}{1+e^x}}}, \; K \in \mathbb{R}$

Aufgabe 5: Gravitation (6 Punkte)

m_S = 1'600kg; $h = 2.4 \cdot 10^6$m;
Fundamentum S. 108: $r_E = 6.37 \cdot 10^6$m; $m_E = 5.97 \cdot 10^{24}$kg

a) $r = r_E + h = 8.77 \cdot 10^6$m

$$F_{Gr} = G \cdot \frac{m_S \cdot m_E}{r^2} = 8'280\text{N}$$

b) Radialkraft = Gravitationskraft: $m_S \cdot \frac{v^2}{r} = G \cdot \frac{m_S \cdot m_E}{r^2}$ → $v = \sqrt{\frac{G \cdot m_E}{r}} = 6'740 \frac{\text{m}}{\text{s}}$

c)

$$\Delta E = E_{Umlaufbahn} - E_{Erdoberfläche} = \frac{m_S \cdot v^2}{2} - G \cdot \frac{m_S \cdot m_E}{r} + G \cdot \frac{m_S \cdot m_E}{r_E}$$

$$= m_S \left(\frac{v^2}{2} - G \cdot m_E \cdot \left[\frac{1}{r} - \frac{1}{r_E} \right] \right) = 6.37 \cdot 10^{10}\text{ J}$$

Aufgabe 6: Strömungen (4 Punkte)

v_1 = 3.5m/s; p_1 = 240'000Pa; d_1 = 0.05m; d_2 = 0.03m; h = 3.5m;
$\rho_W = 1000 \frac{\text{kg}}{\text{m}^3}$

Kontinuitätsgleichung: $A_1 \cdot v_1 = A_2 \cdot v_2$ → $v_2 = \frac{A_1}{A_2} \cdot v_1 = \frac{d_1^2}{d_2^2} \cdot v_1 = 9.72 \frac{\text{m}}{\text{s}}$

Bernoulli: $p_1 + \frac{\rho_W \cdot v_1^2}{2} + \rho_W \cdot g \cdot h = p_2 + \frac{\rho_W \cdot v_2^2}{2}$

→ $p_2 = p_1 + \rho_W \cdot g \cdot h + \frac{\rho_W \cdot \left(v_1^2 - v_2^2\right)}{2} = 233'000\text{Pa}$

Aufgabe 7: Differentialgleichungen, Wechselstromkreise (11 Punkte)

$C = 5\mu$F; $R = 2'000\Omega$

a) U_0 = 12V
Maschenregel: $U_0 = U_R + U_C$
→ $U_0 = R \cdot I + \frac{Q}{C} = R \cdot \dot{Q} + \frac{Q}{C}$
→ Differentialgleichung für $Q(t)$: $\dot{Q} = -\frac{Q}{R \cdot C} + \frac{U_0}{R}$
→ allgemeine Lösung: $Q(t) = Q^* \cdot e^{-\frac{t}{R \cdot C}} + C \cdot U_0$, Integrationskonstante: Q^*
Anfangsbedingungen: $Q(0) = 0$ → $Q^* = -C \cdot U_0$
→ Lösung: $Q(t) = C \cdot U_0 \cdot \left(1 - e^{-\frac{t}{R \cdot C}}\right)$

$$I(t)=\dot{Q}(t)=C\cdot U_0\cdot\left(-\frac{1}{R\cdot C}\right)\cdot\left(-e^{-\frac{t}{R\cdot C}}\right)=\frac{U_0}{R}\cdot e^{-\frac{t}{R\cdot C}}$$

Einsetzen der Zahlenwerte liefert:

$Q(0.02\text{s})=5.19\cdot 10^{-5}\text{C}$; $U(0.02\text{s})=\dfrac{Q(0.02\text{s})}{C}=10.4\text{V}$;

$I(0.02\text{s})=8.12\cdot 10^{-4}\text{A}=0.812\text{mA}$

b) $\hat{U}=12\text{V}$; $\omega=200\text{s}^{-1}$

Gesamtimpedanz der Schaltung: $Z=Z_R+Z_C=R+\dfrac{1}{i\cdot\omega\cdot C}$

$$\frac{\hat{U}}{\hat{I}}=|Z|=\sqrt{R^2+\frac{1}{(\omega\cdot C)^2}}=2'240\Omega$$

$$\rightarrow \hat{I}=\frac{\hat{U}}{|Z|}=5.37\cdot 10^{-3}\text{A}=5.37\text{mA}$$

Aufgabe 8: Kernphysik (9 Punkte)

a) Fundamentum S. 106: $m_{I\text{-}131}$ = 130.906u
$m_{H\text{-}1}$ = 1.0078u; m_n = $1.675\cdot 10^{-27}$kg = 1.0087u
Massendefekt: $\Delta m=53\cdot m_{H-1}+78\cdot m_n-m_{I-131}=1.19\text{u}=1.97\cdot 10^{-27}\text{kg}$
$E_B=\Delta m\cdot c^2=1.77\cdot 10^{-10}\text{J}=1'100\text{MeV}$
Pro Nukleon: $\dfrac{E_B}{A}=8.43\text{MeV}$

b) Fundamentum S. 106: I-131 macht einen β^--Zerfall
${}^{131}_{53}I\rightarrow{}^{131}_{54}Xe+e^-+\nu$, es entsteht Xenon-131.

c) A= 500Bq; $T_{1/2}$ = 8.04d = $6.95\cdot 10^5$s (Fundamentum S. 106)
Zerfallskonstante: $\lambda=\dfrac{\ln 2}{T_{1/2}}=9.98\cdot 10^{-7}\text{s}^{-1}$

$A=\lambda\cdot N \rightarrow N=\dfrac{A}{\lambda}=5.01\cdot 10^{8}$

$m=N\cdot m_{I-131}=1.08\cdot 10^{-16}\text{kg}$

d) A_0 = 2'300Bq; A = 500Bq
Zerfallsgesetz: $A=A_0\cdot e^{-\lambda\cdot t}$

$$\rightarrow \frac{A}{A_0}=e^{-\lambda\cdot t}\rightarrow t=-\frac{1}{\lambda}\cdot\ln\left(\frac{A}{A_0}\right)=1.53\cdot 10^{6}\text{s}=17.7\text{d}$$