Schriftliche Maturitätsprüfung 2013

Kantonsschule Reussbühl

Schwerpunktfach Physik und Anwendungen der Mathematik

Lösungen

Aufgabe 1: (7 Punkte)

$E_{Kin} + E_{Rot} = E_{Pot}$ 1P

$\frac{1}{2} m v_0^2 + \frac{1}{2} I \omega^2 = mgh$

$\Rightarrow \frac{1}{2} m \omega^2 r^2 + \frac{1}{2} I \omega^2 = mgh$ 1P

Vollzylinder:

$I_{Vz} = \frac{1}{2} m r^2$

$\frac{1}{2}\omega^2 r^2 + \frac{1}{4}\omega^2 r^2 = g \cdot h$

$h = \frac{3}{4}\frac{\omega^2 r^2}{g}$ 2P

Kugel:

$I_K = \frac{2}{5} m r^2$

$\frac{1}{2}\omega^2 r^2 + \frac{1}{5}\omega^2 r^2 = g \cdot h$

$h = \frac{7}{10}\frac{\omega^2 r^2}{g}$ 2P

Der Vollzylinder rollt höher hinauf. 1P

Aufgabe 2: (7 Punkte = 4 Punkte + 3 Punkte)

a) $m \cdot a = m \cdot \frac{dv}{dt} = -\beta \cdot v - F_M$

$\Rightarrow \frac{dv}{dt} + \frac{\beta}{m} \cdot v = -\frac{F_M}{m}$ 1P

Die homogene DGL $\frac{dv}{dt} + \frac{\beta}{m} \cdot v = 0$ hat die Lösung $v(t) = Ce^{-\frac{\beta}{m}t}$. 1P

Für ($t \to \infty$) folgt $v_{max} = -F_M/\beta$, das ist die Lösung für die inhomogene DGL.

Die DGL hat nun die allgemeine Lösung $v(t) = Ce^{-\frac{\beta}{m}t} - \frac{F_M}{\beta}$ 1P

mit $v(0) = v_0$ folgt $v_0 = C - \frac{F_M}{\beta} \Rightarrow C = v_0 + \frac{F_M}{\beta}$

und $v(t) = \left(v_0 + \frac{F_M}{\beta}\right) \cdot e^{-\frac{\beta}{m} t} - \frac{F_M}{\beta}$ 1P

b)

$s(t) = \int v(t)\,dt = -\frac{m}{\beta}\left(v_0 + \frac{F_M}{\beta}\right) \cdot e^{-\frac{\beta}{m}t} - \frac{F_M}{\beta} \cdot t + C$ 1P

mit $s(0) = 0$ folgt

$s(0) = -\frac{m}{\beta}\left(v_0 + \frac{F_M}{\beta}\right) + C = 0$

$\Rightarrow C = \frac{m}{\beta}\left(v_0 + \frac{F_M}{\beta}\right)$ 2P

Aufgabe 3: (4 Punkte)

$\Delta t = \frac{\Delta s}{v} \quad (= 2'000\text{s})$ 1P

$E_{mech} = P \cdot \Delta t = P \cdot \frac{\Delta s}{v} \quad (= 1.5 \cdot 10^8\,\text{J})$ 1P

$E_{chem} = H \cdot m = H \cdot \rho_{Benzin} \cdot V \quad (= 4.1 \cdot 10^8\,\text{J})$ 1P

$\eta = \frac{E_{mech}}{E_{chem}} = \frac{P \cdot \Delta s}{v \cdot H \cdot \rho_{Benzin} \cdot V} = 0.37$ 1P

$\Rightarrow \eta = 37\%$

Aufgabe 4: (5 Punkte)

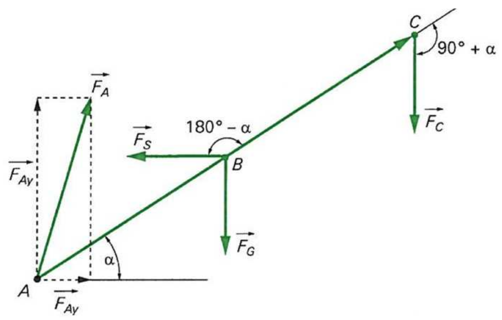
$Z = \frac{U_{eff}}{I_{eff}} = 45.98\Omega \approx 46.0\Omega$ 1P

$R_{Ers} = R_0 + R_1 = \frac{U_G}{I_G} \Rightarrow R_0 = \frac{U_G}{I_G} - R_1 = 11.96\Omega \approx 12.0\Omega$ 2P

$Z = \sqrt{R_{ers}^2 + (\omega \cdot L)^2} \Rightarrow L = \frac{1}{2\pi \cdot f}\sqrt{Z^2 - R_{ers}^2} \approx 35.2\text{mH}$ 2P

Aufgabe 5: (7 Punkte = 4 Punkte + 3 Punkte)

a) Skizze (nicht gefragt):



$l \cdot F_C \cos\alpha + \frac{1}{2} l \cdot F_G \cos\alpha - \frac{1}{2} F_S \sin\alpha = 0$

$2F_C \cos\alpha + F_G \cos\alpha - F_S \sin\alpha = 0$ 1P

$F_C = \frac{F_S \sin\alpha - F_G \cos\alpha}{2\cos\alpha}$ 1P

$\sin\alpha = \frac{h}{l/2} = \frac{2h}{l} \Rightarrow \alpha = 32.2^\circ$ 1P

$\Rightarrow F_C = 1629\ \text{N}$ 1P

b)

$F_{AX} = F_S = 7500\ \text{N}$

$F_{AY} = F_G + F_C = 3100\ \text{N}$ 3P

$F_A = \sqrt{F_{AX}^2 + F_{AY}^2} = 8115\ \text{N}$

Aufgabe 6: Raumgeometrie (8 Punkte)

a) Normalvektor zu E: $\vec{n}_E = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$,

Normalgerade durch M: $n: \vec{r} = \vec{r}_M + t\vec{n}_E = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$E \cap n$: $(2+t) - 2(3-2t) + 2(4+2t) + 23 = 0 \Rightarrow 9t + 27 = 0 \Rightarrow t = -3$

$\Rightarrow \vec{r}_Q = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ -2 \end{pmatrix} \Rightarrow \underline{\underline{Q(-1|9|-2)}}$

$\Rightarrow d = \overline{MQ} = \sqrt{(-1-2)^2 + (9-3)^2 + (-2-4)^2} = \sqrt{81} = 9$

$\Rightarrow r = \sqrt{R^2 - d^2} = \sqrt{15^2 - 9^2} = \sqrt{144} \Rightarrow \underline{\underline{r = 12}}$

b) Gleichung von K: $(x-2)^2 + (y-3)^2 + (z-4)^2 = 15^2 = 225$

$K \cap n$: $((2+t)-2)^2 + ((3-2t)-3)^2 + ((4+2t)-4)^2 = 225$

$\Rightarrow t^2 + (-2t)^2 + (2t)^2 = 225 \Rightarrow 9t^2 = 225 \Rightarrow t_{1/2} = \pm 5$.

$t_2 = -5$ liegt näher bei $t = -3$: $\vec{r}_B = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - 5 \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 13 \\ -6 \end{pmatrix} \Rightarrow B(-3|13|-6)$

Ans. für Gl. von $T: x - 2y + 2z + k = 0$, $B \in T \Rightarrow -3 - 26 - 12 + k = 0 \Rightarrow k = 41$

$\Rightarrow \underline{\underline{T: x - 2y + 2z + 41 = 0}}$

c) Höhe des Kegels und des Kugelsegments: $h = R - d = 15 - 9 = 6$,

$\underline{\underline{V_{Kegel}}} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 12^2 \cdot 6 \underline{\underline{= 288\pi}}$, $\underline{\underline{V_{Segm.}}} = \frac{1}{3}\pi h^2(3R - h) = \frac{1}{3}\pi \cdot 36 \cdot (3 \cdot 15 - 6) \underline{\underline{= 468\pi}}$

$\underline{\underline{\frac{V_{Kegel}}{V_{Segm.}}}} = \frac{288\pi}{432\pi} = \frac{8}{13} \underline{\underline{\cong 61.54\%}}$

Aufgabe 7: Komplexe Funktionen (8 Punkte)

a) Nullstellen: $h(z) = -3i \cdot z^2 - 2i = 0 \Rightarrow z^2 = -\frac{2}{3} \Rightarrow \underline{\underline{z_{1/2}}} = \pm\sqrt{\frac{2}{3}}i = \underline{\underline{\pm\frac{\sqrt{6}}{3}i}}$

Fixpunkte: $h(z) = -3i \cdot z^2 - 2i = z \Rightarrow -3i \cdot z^2 - z - 2i = 0 \Rightarrow z^2 - \frac{1}{3}iz + \frac{2}{3} = 0$

$D = (-\frac{1}{3}i)^2 - 4 \cdot \frac{2}{3} = -\frac{1}{9} - \frac{8}{3} = -\frac{25}{9} = w^2 \Rightarrow w = \pm\frac{5}{3}i \Rightarrow z_{3/4} = \frac{\frac{1}{3}i \pm \frac{5}{3}i}{2} = \frac{(1\pm 5)}{6}i$

$\Rightarrow \underline{\underline{z_3 = -\frac{2}{3}i}}, \ \underline{\underline{z_4 = i}}$

b) Schnittstellen: $\frac{9}{z} - 2i = -3i \cdot z^2 - 2i \Rightarrow 9 = -3iz^3 \Rightarrow z^3 = 3i = 3e^{i\frac{\pi}{2}} \Rightarrow$

$\underline{\underline{z_5}} = \sqrt[3]{3}e^{i\frac{\pi}{6}} = \sqrt[3]{3}(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})) = \sqrt[3]{3}(\frac{\sqrt{3}}{2} + \frac{1}{2}i) = \underline{\underline{\frac{\sqrt[3]{3}\sqrt{3}}{2} + \frac{\sqrt[3]{3}}{2}i}}$

$\underline{\underline{z_6}} = \sqrt[3]{3}e^{i(\frac{\pi}{6}+\frac{2\pi}{3})} = \sqrt[3]{3}e^{i\frac{5}{6}\pi} = \sqrt[3]{3}(\cos(\frac{5}{6}\pi) + i\sin(\frac{5}{6}\pi)) = \sqrt[3]{3}(-\frac{\sqrt{3}}{2} + \frac{1}{2}i) = \underline{\underline{-\frac{\sqrt[3]{3}\sqrt{3}}{2} + \frac{\sqrt[3]{3}}{2}i}}$

$\underline{\underline{z_7}} = \sqrt[3]{3}e^{i(\frac{\pi}{6}+\frac{4\pi}{3})} = \sqrt[3]{3}e^{i\frac{3}{2}\pi} = \sqrt[3]{3}(-i) = \underline{\underline{-\sqrt[3]{3}i}}$

c) $k: |w| = 1 \Rightarrow w\overline{w} = 1, \ w = f(z) = \frac{9}{z} - 2i, \ \overline{w} = \frac{9}{\overline{z}} + 2i$

$f^{-1}(k): (\frac{9}{z} - 2i)(\frac{9}{\overline{z}} + 2i) = 1 \Rightarrow \frac{81}{z\overline{z}} - \frac{18i}{\overline{z}} + \frac{18i}{z} + 4 = 1 \Rightarrow 81 - 18iz + 18i\overline{z} + 3z\overline{z} = 0$

$\Rightarrow z\overline{z} + 6i\overline{z} - 6iz + 27 = 0 \Rightarrow (z+6i)(\overline{z} - 6i) - 36 + 27 = 0 \Rightarrow |z+6i|^2 = 9 \Rightarrow \underline{\underline{|z-(-6i)| = 3}}$

Urbild von k auch Kreis mit Mittelpunkt $m = -6i$ und Radius $r = 3$

Aufgabe 8: Differentialgleichung (8 Punkte)

a) $y'' + 4y' + \frac{25}{4}y = 0 \Rightarrow$ charakt. Gl. $\lambda^2 + 4\lambda + \frac{25}{4} = 0 \Rightarrow \lambda_{1/2} = \frac{-4 \pm \sqrt{16-25}}{2} = -2 \pm \frac{3}{2}i$

$\Rightarrow \underline{\underline{y_H = e^{-2x}(C_1\sin(\frac{3}{2}x) + C_2\cos(\frac{3}{2}x))}}$

b) Ansatz: $y_p = k_1x + k_2 \ (y_p \notin y_H) \Rightarrow y'_p = k_1, \ y''_p = 0$

$\overset{DGL}{\Rightarrow} 0 + 4k_1 + \frac{25}{4}(k_1x + k_2) = 25x + 16 \Rightarrow \frac{25}{4}k_1x + 4k_1 + \frac{25}{4}k_2 = 25x + 16$

$\Rightarrow \left| \begin{array}{l} \frac{25}{4}k_1 = 25 \\ 4k_1 + \frac{25}{4}k_2 = 16 \end{array} \right| \Rightarrow \left| \begin{array}{c} k_1 = 4 \\ 16 + \frac{25}{4}k_2 = 16 \end{array} \right| \Rightarrow \left| \begin{array}{l} k_1 = 4 \\ k_2 = 0 \end{array} \right| \Rightarrow \underline{\underline{y_p = 4x}}$

$\Rightarrow \underline{\underline{y}} = y_H + y_p = \underline{\underline{e^{-2x}(C_1\sin(\frac{3}{2}x) + C_2\cos(\frac{3}{2}x)) + 4x}}$

c) $\left| \begin{array}{l} y(0) = 0 \\ y(\pi) = 0 \end{array} \right| \Rightarrow \left| \begin{array}{l} 0 = e^0 \cdot (C_1\sin(0) + C_2\cos(0)) + 0 = C_2 \\ 0 = e^{-2\pi}(C_1\sin(\frac{3}{2}\pi) + C_2\cos(\frac{3}{2}\pi)) + 4\pi = e^{-2\pi}(C_1(-1) + 0) + 4\pi \end{array} \right| \Rightarrow \left| \begin{array}{l} C_2 = 0 \\ C_1 = 4\pi e^{2\pi} \end{array} \right|$

$\underline{\underline{y_s}} = e^{-2x}(4\pi e^{2\pi}\sin(\frac{3}{2}x)) + 4x = \underline{\underline{4\pi e^{2(\pi-x)}\sin(\frac{3}{2}x) + 4x}}$

Aufgabe 9:

a) **Affine Abbildungen (3 Punkte)**

$$\underline{\underline{\sigma(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\vec{x}}},\quad \underline{\underline{\rho(\vec{x})}} = \begin{pmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{pmatrix}\vec{x} = \underline{\underline{\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\vec{x}}}$$

$$\Rightarrow \underline{\underline{\alpha(\vec{x})}} = \rho(\sigma(\vec{x})) + \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{\sqrt{2}}{2} & \sqrt{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}}}$$

b) **Rotationsvolumen (3 Punkte)**

$$f(x) = \frac{\ln(x)}{\sqrt{x}} \;\Rightarrow\; V = \pi\int_1^{e^3} \left(\frac{\ln(x)}{\sqrt{x}}\right)^2 dx = \pi\int_1^{e^3} \frac{\ln^2(x)}{x}\,dx$$

$\int \frac{\ln^2(x)}{x}\,dx$; Substitution: $u = \ln(x) \;\Rightarrow\; \frac{du}{dx} = \frac{1}{x} \;\Rightarrow\; du = \frac{1}{x}dx$

$$\Rightarrow \int \frac{\ln^2(x)}{x}\,dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\ln^3(x)}{3} + C, C \in IR$$

$$\Rightarrow \underline{\underline{V}} = \pi\int_1^{e^3} \frac{\ln^2(x)}{x}\,dx = \pi\left[\frac{\ln^3(x)}{3}\right]_1^{e^3} = \pi\left(\frac{3^3}{3} - 0\right) \underline{\underline{= 9\pi}}$$